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On the removal of some ambiguities in elastic amplitude analysis

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Abstract. We show that in deducing amplitudes from cross sections for spinless particles the ambiguities associated with the complex conjugation of a zero in the $\cos\theta$ plane are excluded by imposing the correct nearby branch-cut structure. In addition, we find that these same properties rule out the possibility of forming amplitudes with an infinite number of partial waves but giving rise to polynomial cross sections.

1. Introduction

Phase-shift analysis continues to be an important technique in deducing information on partial wave amplitudes from differential cross section measurements in scattering processes. In the past it was often supposed that the more accurately measurements were made, the more accurately could the phase shifts be determined and that, in the case of perfect measurements, a unique scattering amplitude could be obtained. It is indeed true that when we are dealing with the scattering of spinless particles from spinless particles at an energy where no other process than elastic scattering is allowed, it is possible (Martin 1969) to deduce sufficient conditions on the cross section for it to correspond to a unique amplitude. However, these conditions are extremely restrictive and may only hold very close to zero scattering energy in most practical cases.

When they are not met, a number of explicit examples have been constructed which show a two-fold ambiguity. This means that it was possible to construct two different amplitudes $f_1(\cos\theta)$ and $f_2(\cos\theta)$ such that

$$|f_1(\cos\theta)|^2 = |f_2(\cos\theta)|^2 = k^2(\mathrm{d}\sigma/\mathrm{d}\Omega)(\cos\theta) \qquad \text{for } -1 \leqslant \cos\theta \leqslant 1,$$

$f_1(\cos\theta)$ and $f_2(\cos\theta)$ also both satisfy the elastic unitarity relation

$$\mathrm{Im}\, \mathrm{f}(cos\, \theta_{12}) = \frac{1}{4\pi}\int f^*(\cos\theta_{13}) f(\cos\theta_{32})\, \mathrm{d}\Omega_3.$$

This is quite apart from and in addition to the so-called 'trivial' ambiguity $f_2(\cos\theta) = -f_1^*(\cos\theta)$, which just involves reversing the sign of all phase shifts, and which plays no part in the following.

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The first example to exhibit a two-fold ambiguity was constructed by Crichton (1966). His amplitudes contained only S, P and D waves and may be written in the form

$$\tfrac{15}{2}\exp(\mathrm{i}\delta_2)\sin\delta_2(\cos\theta - x_1(\delta_2)\pm \mathrm{i}y(\delta_2))(\cos\theta - \tfrac{4}{5}+\tfrac{1}{5}\,\mathrm{i}\cot\delta_2),$$

polynomials in $\cos\theta$ which are related by the complex conjugation of one root. Then two-fold ambiguities were constructed with S, P, D and F waves and with S, P, D, F and G waves. Berends and van Reisen (1976) have found examples of non-unique amplitudes in which an arbitrary but finite number of partial waves are present. It has also been shown that a two-fold ambiguity was the maximal possible uncertainty in the case of an entire, non-polynomial amplitude (Itzykson and Martin 1973).

In all these cases the equivalent amplitudes are formed by replacing one or more zeros of an amplitude in the complex $\cos\theta$ plane by its complex conjugate; for brevity we shall refer to these as Crichton amplitudes. Replacing a zero by its complex conjugate would in general give rise to an amplitude which is non-unitary; the sepcial feature of the examples constructed is that the phase shifts are so chosen that the complex conjugation also results in a unitary amplitude.

However, the main interest of these investigations is in the application to actual elastic-region phase-shift analyses and one can raise doubts about the relevance of these artificially constructed amplitudes to real processes. For example the phase shifts δ_l constructed by Berends and van Reisen (1976) have the property that

$$\delta_{l+2}-\delta_l = \text{constant for } l = 0, 2, \ldots, L-2.$$

Since, for any short-range potential, δ_l tends to zero for large l, this condition cannot be met by a physical amplitude. More generally, the wave nature of particles means that an infinite number of partial waves are affected by the interaction. Indeed if one considers the scattering amplitude, f, as a function of complex $\cos\theta = z$ then $f(z)$ is known to have branch points along the real axis (Bowcock and Burkhardt 1975). Polynomials clearly do not possess such singularities. Thus one may wonder whether there is a Crichton ambiguity at all when the partial wave series is infinite and has a finite ellipse of convergence. In fact Atkinson *et al* (1977) demonstrated that it is possible to construct unitary amplitudes with an infinite number of partial waves which fall off exponentially as the angular momentum tends to infinity, and for which the ambiguity still exists. Nevertheless, what these amplitudes do not possess is a behaviour near to the branch points which corresponds to a physical amplitude.

The cut structure which we shall employ and which is well founded in theory is that $f(z)$ is an analytic function of z in the whole complex z plane with the exception of singularities along the real axis. Typically, the singularity nearest to the physical region $-1 \leqslant z \leqslant 1$ is a pole at $z = x_{\mathrm{p}}$, say, with real residue followed by branch points at $x_0, x_1, \ldots$ where $1 < x_{\mathrm{p}} \leqslant x_0 \leqslant x_1$. Unitarity requires that the imaginary part of the amplitude has a larger region of analyticity than the real part and that when $x_0 < x < x_1$ the discontinuity across the branch cut must bė purely imaginary, i.e.

$$\mathrm{Re}[f(x+\mathrm{i}\varepsilon)-f(x-\mathrm{i}\varepsilon)] = 0 \qquad \text{for } x_0 < x < x_1. \tag{1}$$

The requirement that an amplitude possess this analytic structure immediately rules out of course any polynomial amplitudes, as these are entire functions. However, we shall see in the next section that Crichton ambiguities with an infinite number of partial waves are also eliminated. We shall also show that an extension of the requirement (1) may be used to eliminate polynomial cross sections arising from non-polynomial amplitudes.

2. Complex conjugation of zeros

The equivalent unitary amplitudes under discussion are related by the complex conjugation of a zero of an amplitude. If $f_1(z)$ has a zero at $z = z_1$ which is complex, we may write

$$f_1(z) = [(z - z_1)/(1 - z_1)]g(z)$$

and conjugation of z_1 gives an amplitude

$$f_2(z) = [(z - z_1^*)/(1 - z_1^*)]g(z)$$

which has the same modulus for $-1 \leqslant z \leqslant 1$. The additional factor $1 - z_1$ introduced into the denominator ensures that $\operatorname{Im} f_1(1) = \operatorname{Im} f_2(1)$ and hence the total cross section remains unchanged. $g(z)$ will contain the pole and branch points on the real axis.

If we now impose condition (1) on $f_1(z)$ we obtain

$$\operatorname{Re}\{[(z - z_1)/(1 - z_1)][g(x + i\varepsilon) - g(x - i\varepsilon)]\} = 0 \qquad \text{for } x_0 < x < x_1.$$

Setting $z_1 = \alpha + i\beta$ and $g(x + i\varepsilon) - g(x - i\varepsilon) = \rho(x) = \rho_1(x) + i\rho_2(x)$, where $\alpha, \beta, \rho_1, \rho_2$ are all real, this gives

$$[(x - \alpha)(1 - \alpha) + \beta^2]\rho_1(x) - \beta(x - 1)\rho_2(x) = 0. \tag{2}$$

Requiring the complex conjugated amplitude, f_2, to possess the same analytic structure and also satisfy (1) gives the same equation but with the sign of β reversed

$$[(x - \alpha)(1 - \alpha) + \beta^2]\rho_1(x) + \beta(x - 1)\rho_2(x) = 0. \tag{3}$$

Since $\beta \neq 0$ (2) and (3) can be satisfied only if $\rho_1(x) = \rho_2(x) = 0$. But this means that $f_1(z)$ and $f_2(z)$ have no discontinuity across the branch cut which is inconsistent with the required analytic structure.

Thus a Crichton ambiguity, produced through complex conjugation of a zero, is impossible for amplitudes with correct branch-cut structure. A practical consequence of this should be that in any technique for deducing an amplitude from data on the cross section, a parameterisation should be used which incorporates this structure explicitly. Note that the presence or absence of a pole in $f(z)$ does not affect the above argument. The very high partial waves would be dominated by such a pole and the results of Atkinson *et al* (1977) show that specification of the high partial waves is not sufficient in general to remove the two-fold ambiguity. Our result is quite consistent with this.

3. Removal of additional ambiguities

There is another type of ambiguity which corresponds to a transformation more general than complex conjugation and to which the above argument is therefore not applicable. This new kind of ambiguity involves starting from two Crichton amplitudes $f_1(z)$ and $f_2(z)$ for which $\sigma = |f_1(z)|^2 = |f_2(z)|^2$ and continuing away from these by adding to the cross section additional polynomial terms of higher order to give $\sigma + \delta\sigma$. Thus if f_1 and f_2 contain only S, P and D waves, σ would be a fourth-order polynomial in z and $\sigma + \delta\sigma$ a higher polynomial in which $\delta\sigma$ is small. Then corresponding to $\sigma + \delta\sigma$ it has been shown (Atkinson *et al* 1974) that it is possible to construct amplitudes $f_1 + \delta f_1$ and $f_2 + \delta f_2$ which differ infinitesimally from f_1 and f_2 and for which $|f_1 + \delta f_1|^2 =$

$|f_2+\delta f_2|^2=\sigma+\delta\sigma$. In this way one can move continuously away from a polynomial cross section to another polynomial cross section of higher order. This does not mean that the new amplitudes $f_1+\delta f_1$ and $f_2+\delta f_2$ are themselves polynomial amplitudes. If this were the case, they could be discounted as not possessing a branch cut structure, indeed these amplitudes are allowed to have branch points outside the Martin–Lehmann ellipse. The two new amplitudes are not simply related by the complex conjugation of a zero; however, we shall show below that the correct analytic also rules out the possibility of their being acceptable.

Our starting point is to use the expression

$$\sigma=f(z)f^*(z^*) \tag{2}$$

in the physical region where z is real. This expression may now be analytically continued into the complex z plane. We must require $f(z)$ to have the correct analytic structure specified in (1). This means that

$$f(x\pm i\varepsilon)=\alpha(x)+i\beta(x)\pm i\gamma(x) \qquad \text{for } x_0<x<x_1$$

where $\alpha(x), \beta(x)+\gamma(x)$ are all real. Analytically continuing (2) to the upper and lower side of the cut we find

$$\sigma(x+i\varepsilon)-\sigma(x-i\varepsilon)=4i\alpha(x)\gamma(x), \qquad x_0<x<x_1.$$

This equation holds for any amplitude with the correct nearby branch cut. However, in the case under consideration $\sigma+\delta\sigma$ is a polynomial so that there can be no discontinuity of $\sigma+\delta\sigma$ across the real axis if it is continued to $x_0<x<x_1$. Hence $\alpha(x)\gamma(x)=0$.

Since we are insisting that $f(z)+\delta f(z)$ has a branch cut, $\gamma(x)$ cannot be zero and hence $\alpha(x)=0$ for $x_0<x<x_1$ i.e.

$$\alpha(x)=\mathrm{Re}(f_1+\delta f_1)=0 \qquad \text{for } x_0<x<x_1. \tag{3}$$

However, f_1 is a polynomial whose coefficients are of the order of unity. A continuous change in δf_1 from zero clearly cannot satisfy (3).

We may strengthen this argument if use is made of the normal two-particle nature of the first branch point at $z=x_0$, i.e. taking the branch point to be of a square-root type. This dependence may be introduced by using the variable $w=u+iv=(z-x_0)^{1/2}$. In terms of this variable there will be a neighbourhood of x_0 in which $\delta f_1(z)$ may be expanded

$$\delta f_1=\sum_{n=0}^{\infty} a_n w^n,$$

whereas f_1, which is a polynomial in z, may be written

$$f_1=\sum_{n=0}^{L} b_n w^{2n}.$$

Imposing the purely imaginary nature of δf_1 between $z=x_0$ and $z=x_1$ gives

$$\mathrm{Re}\sum_{n=0}^{\infty}[a_n u^n-a_n(-u)^n]=0 \qquad \text{or} \qquad \mathrm{Re}\, a_{2n+1}=0.$$

Now using $\mathrm{Re}(f_1+\delta f_1)=0$ in the same interval gives in addition

$$\mathrm{Re}(a_{2n}+b_{2n})=0 \qquad \text{for } n=0,1,\ldots,L \tag{5}$$

and

$$\mathrm{Re}\, a_{2n} = 0 \qquad \text{for } n > L.$$

Now consider the amplitude $f_1 + \delta f_1$ *inside* the Martin–Lehmann ellipse on the real axis close to the first branch point. This corresponds to $w = \mathrm{i}v$ and

$$f_1 + \delta f_1 = \sum_{n=0}^{\infty} (a_{2n} + b_{2n})(\mathrm{i}v)^{2n} + \sum_{n=0}^{\infty} a_{2n+1}(\mathrm{i}v)^{2n+1}.$$

Using (5) and (6) reduces this to $f_1 + \delta f_1 = \sum_{n=0}^{\infty} a_{2n+1}(\mathrm{i}v)^{2n+1}$, which is purely imaginary. Since there is no other branch point between this part of the z axis and the physical region, analytic continuation leads to a purely imaginary amplitude over the whole range $-1 \leqslant z \leqslant 1$. This violates unitarity.

4. Summary and discussion

What has been shown is that insistence on the correct nearby analytic structure in the complex $\cos\theta$ plane removes both Crichton ambiguities produced by the complex conjugation of a single zero and also eliminates the possibility of unitary amplitudes with infinitely many partial waves giving rise to a polynomial cross section. It will be of interest to investigate the practical consequences of these results in the process of deriving amplitudes from cross sections by using parametrisations which incorporate the required analytic structure. Although in practice we have to work with imprecise experimentally determined cross sections which means that some uncertainty in the determination of amplitudes is unaviodable, nevertheless the large discrete Crichton ambiguities should be removed.

The arguments have been formulated here in terms of analyticity in the complex $\cos\theta$ plane but similar reasoning can be used in mapped variables. In particular, complex conjugation of Barrelet (1972) zeros which is commonly used may also be treated in the same manner.

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